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On Einstein–Podolsky–Rosen paradox

Ramon Carbó-Dorca

Department of Inorganic and Physical Chemistry, Ghent University, Krijgslaan 281, B-9000 Gent, Belgium and Institut de Química Computacional, Universitat de Girona, Girona 17071, Catalonia Spain

The Einstein–Podolsy–Rosen (EPR) paradox is analyzed. Here is shown that, when EPR wavefunctions are submitted to reasonable normalization and the reference frames rotated by unitary transformations, the EPR paradox disapears.

KEY WORDS: Einstein–Podolsky–Rosen paradox

1. Introduction

In 1935, Einstein, Podolsky and Rosen (EPR) [1] plublished a well known paper, headed with a question about a polemic aspect of quantum mechanics, which undoubtely has generated a large amount of literature: whether quantum mechanics presents a *complete* structure, defining such a characteristic in a sufficiently logical manner. As a result of his study, EPR concluded that quantum mechanics was *not* complete. It will be cumbersome to write such a large list of discussions on the EPR paradox contents, so only some relevant literature will be quoted and mainly chosen from the same source as the old EPR paper. Take, as an example of this last remark, a series of theoretical papers due to Bohm [2, 3, 4], a relatively recent EPR related experimental work [5] or a very recent point of view concerning EPR and teleportation [6]. The work of Bell [7] shall be finally mentioned as a source of analysis of all the problems involved in EPR studies, leading to the so-called EPR paradox.

Here, the basic asumption of EPR will be analyzed and some conclusions obtained.

2. The basic EPR wavefunction form

In his 1935 paper, EPR proposed to construct two wavefunction forms for a system made of two particles, which had suffered some kind of interaction in time past, but being in a noninteracting stationary state at the present.

This situation can be easily generalized, for example as Treacy proposes [6], in the present work, however, the EPR original formalism in terms of the following scheme will be used:

(i) Two submicroscopic systems **S(1)** and **S(2)** have interacted in the past and are now in a noninteracting stationary state.

(ii) Two Hermitian operators A and B, acting on system **S(1)** are known.

(iii) The secular equtions of both operators are known and can be written as:

$$Aa_k(\mathbf{x}) = \alpha_k a_k(\mathbf{x})$$

and

$$Bb_k\left(\mathbf{x}\right) = \beta_k b_k\left(\mathbf{x}\right),$$

where:

$$\{\alpha_k\} \wedge \{a_k\left(\mathbf{x}\right)\} \text{ and } \{\beta_k\} \wedge \{b_k\left(\mathbf{x}\right)\} \tag{1}$$

are the eigenvalues and eigenfunctions for the operators A and B, respectively. By $\mathbf{x}$ there are represented the variables describing system **S(1).**

(iv) Two complete sets of functions are related to system **S(2)**, known and writen as: $\{f_k\left(\mathbf{y}\right)\} \wedge \{g_k\left(\mathbf{y}\right)\}$, where the vector $\mathbf{y}$ refers to the variables of the system **S(2)**.

EPR constructed two forms of a wavefunction for the situation of the *two* present noninteracting systems status, namely:

$$\Psi_A\left(\mathbf{x},\mathbf{y}\right) = \sum_k f_k\left(\mathbf{y}\right) a_k\left(\mathbf{x}\right) \tag{2}$$

and

$$\Psi_B\left(\mathbf{x},\mathbf{y}\right) = \sum_k g_k\left(\mathbf{y}\right) b_k\left(\mathbf{x}\right). \tag{3}$$

The construction form of the equivalent functions (2) and (3) is the same as to perform a Hadamard product of two functions, for example:

$$\Psi_A\left(\mathbf{x},\mathbf{y}\right) = F_A\left(\mathbf{y}\right) * \Phi_A\left(\mathbf{x}\right) = \left(\sum_k f_k\left(\mathbf{y}\right)\right) * \left(\sum_k a_k\left(\mathbf{x}\right)\right)$$

and a similar form can be attached to the B reference frame.

EPR did not deeply examine the implications of this construction, excepting its use to prove a possible basic faulty structure in quantum mechanics, when performing a measure. Some similar work on the problem has been discussed by Bell [7] too.

3. Normalization consequences of the EPR wavefunction

However, both function forms as written above, in the case they can be used in quantum mechanics for any purpose, shall be normalized [8]; that is, the following equation must hold for both wavefunction representations:

$$\int_{D_y}\int_{D_x} |\Psi_I|^2 \,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y} = 1 \wedge I = A, B,$$

where the integration in every system variables is performed over some defined domain: D_x, D_y, respectively. That means for the first wavefunction form: Ψ_A, that:

$$1 = \sum_k \sum_l \left(\int_{D_y} f_k^* f_l \,\mathrm{d}\mathbf{y} \right) \left(\int_{D_x} a_k^* a_l \,\mathrm{d}\mathbf{x} \right) = \sum_k \int_{D_y} |f_k|^2 \,\mathrm{d}\mathbf{y} = \sum_k \theta_k^A$$

as the set of eigenfunctions $\{a_k(\mathbf{x})\}$ of the Hermitian operator A is an orthonormalized one.

No wonder, one arrives to a well known result in quantum mechanics: the squared modules of the coordinates with respect the basis set of the eigenfunctions of the operator A constitute a probability distribution, which will be noted as: $\left\{\theta_k^A\right\}$. The same holds for the second wavefunction form Ψ_B, associated to the Hermitian operator B, which can be briefly written:

$$1 = \sum_k \int_{D_y} |g_k|^2 \mathrm{d}\mathbf{y} = \sum_k \theta_k^B,$$

implying that the set of coefficients $\left\{\theta_k^B\right\}$ is a probability distribution too.

Normalization of the EPR wavefunction representation in different orthonormalized basis sets precludes, then, that the sum of the sets of norms of the coefficient functions, associated to the system **S(2)**, shall converge and, thus, transform into a pair of probability distributions; or if the reader prefers it: into two convex sets of coefficients.

If the EPR wavefunctions cannot be normalized, then nothing can be further said from the quantum mechanical point of view. Therefore, if this is the case the EPR result could not be taken as a statement affecting quantum mechanics.

4. Equivalence between EPR wavefunction forms

In order to be able to follow EPR reasoning further, thus, the sets of system **S(2)** functions will be considered from now on as submitted to the consequences of the corresponding wavefunction normalization.

Moreover, both employed eigenfunction sets as described in equation (1) not only are orthonormal, being the eigenfunctions of a Hermitian operator, but they shall be related by a unitary transformation like:

$$\mathbf{U}\mathbf{U}^{+} = \mathbf{U}^{+}\mathbf{U} \wedge \mathbf{U} = \{u_{kl}\} \rightarrow \delta_{lk} = \sum_{p} u_{lp} u_{pk}^{*} = \sum_{p} u_{kp}^{*} u_{pl} = \delta_{kl},$$

which can be interpreted as a rotation in the Hilbert infinite-dimensional space, where both reference frames belong. This is the same to say that the following transformations between both basis sets shall hold:

$$\forall k : b_k = \sum_{l} u_{lk} a_l \wedge a_k = \sum_{l} u_{kl}^{*} b_l. \tag{4}$$

This basis set relationships will provide the equivalence between both wavefunction representations, where from now on the function variables will be dropped to easy the notation, as:

$$\begin{aligned}\Psi_A(\mathbf{x}, \mathbf{y}) &= \sum_{k} f_k\, a_k = \sum_{k} f_k \left(\sum_{l} u_{kl}^{*} b_l \right) \\ &= \sum_{l} \left(\sum_{k} u_{kl}^{*} f_k \right) b_l = \sum_{l} g_l b_l = \Psi_B(\mathbf{x}, \mathbf{y})\end{aligned}$$

a result which furnishes a well known property of vectors in linear algebra, relating the coordinates of the EPR wavefunction by means of the unitary transformation:

$$\forall l : g_l = \sum_{k} u_{kl}^{*} f_k$$

and, on the other hand, one can easily deduce that:

$$\forall k : f_k = \sum_{l} u_{lk} g_l$$

also holds.

One arrives to the trivial result that both wavefunction representations describe the same quantum mechanical object.

Therefore, any EPR normalized wavefunction constructed in the same way, that is, with the orthonormalized eigenfunctions of a non commuting Hermitian operator, acting over system **S(1)**, and a complete set of functions associated to system **S(2)**, used as EPR wavefunction coordinate coefficients, will produce the *same* EPR wavefunction.

5. Expectation values of EPR wavefunctions

According to quantum mechanical lore [8], the expectation values of operator A or B with respect the EPR wavefunction shall be computed by means of the usual formalism.

So, employing Dirac notation to simplify the involved integral expressions and dropping the subscript of the EPR wavefunction, it can be written:

$$\begin{aligned}\langle\Psi|A|\Psi\rangle &= \sum_k\sum_l \langle f_k|f_l\rangle\,\langle a_k|A|a_l\rangle = \sum_k\sum_l \langle f_k|f_l\rangle\alpha_l\,\langle a_k|a_l\rangle \\ &= \sum_k\sum_l \langle f_k|f_l\rangle\alpha_l\delta_{kl} = \sum_k \alpha_l\,\langle f_k|f_k\rangle = \sum_k \alpha_k\theta_k^A = \langle A\rangle\end{aligned}$$

and the equivalent result for the expectation values of operator B can be found to be expressible as:

$$\langle\Psi|B|\Psi\rangle = \sum_k \beta_k\theta_k^B \; = \; \langle B\rangle\,.$$

Note that due to the unitary relationship (4) between both EPR wavefunction representations, the expectation values of a given operator becomes the same in both basis sets; for instance:

$$\begin{aligned}\langle\Psi_A|B|\Psi_A\rangle &= \sum_k\sum_l \langle f_k|f_l\rangle\,\langle a_k|\,B\,|a_l\rangle \\ &= \sum_k\sum_l \langle f_k|f_l\rangle\Big\langle\sum_p u_{kp}b_p|B|\sum_q u_{lq}^* b_q\Big\rangle \\ &= \sum_k\sum_l \langle f_k|f_l\rangle\sum_p u_{kp}\sum_q u_{lq}^*\,\langle b_p|B|b_q\rangle \\ &= \sum_k\sum_l \langle f_k|f_l\rangle\sum_p u_{kp}u_{lp}^*\beta_p = \sum_p\beta_p\sum_k\sum_l u_{kp}\,\langle f_k|f_l\rangle u_{lp}^* \\ &= \sum_p \beta_p\,\langle g_p|g_p\rangle = \sum_p \beta_p\theta_p^B = \langle B\rangle = \langle\Psi_B|\,B\,|\Psi_B\rangle\,.\end{aligned}$$

A result consistent with the fact pointed out earlier that: $\Psi_A = \Psi_B$.

Thus any EPR wavefunction, represented in an adequate orthonormalized framework of the eigenfunctions of noncommuting Hermitian operators, can be used to assess the measure of an expectation value of a Hermitian noncommuting operator.

The EPR argument leading to the paradox was based into the fact that the possible outcome from an experiment attached to the observable associated to operator B, say:

$$\langle B\rangle = \beta_r,$$

will produce two kinds of attached wavefunctions in system **S(2)**. But taking into account the discussed results up to now, one can say that this it is not the case, whenever the EPR wavefunction is normalized.

The paradox do not appears in the present description, because the system **S(2)** functions are simple coefficients, as EPR described them, but submitted to unitary transformations when measuring A or B in their corresponding orthonormalized reference frames. The measure of any operator can be performed within its proper orthonormalized eigenfunction framework or *in any other* orthonormalized basis set, associated to any noncommuting Hermitian operator.

Moreover, the outcome of the measure:

$$\langle B \rangle = \beta_r = \sum_k \beta_k \theta_k^B$$

is perfectly admissible within the quantum mechanical context described in the present paper and just implies:

$$\beta_r = \beta_r \theta_r^B + \sum_{k \neq r} \beta_k \theta_k^B \rightarrow \sum_{k \neq r} (\beta_k - \beta_r) \theta_k^B = 0 \tag{5}$$

as, obviously, being the set $\left\{\theta_k^B\right\}$ convex one can use:

$$1 - \sum_{k \neq r} \theta_k^B = \theta_r^B$$

leading to the result of equation (5).

Furthermore, without loss of generality one can suppose that the eigenvalues of both operators A and B are ordered in ascending fashion, that is:

$$\beta_0 \leqslant \beta_1 \leqslant \cdots \leqslant \beta_r \leqslant \cdots$$

then, using the relationship (5) one can write an equality between two positive definite sums, like:

$$\sum_{k > r} (\beta_k - \beta_r) \theta_k^B = \sum_{k < r} |\beta_k - \beta_r| \theta_k^B .$$

This result is the specific condition which must hold, when using an EPR wavefunction, in order that it is obtained as an expectation value of the operator B one of its eigenvalues: β_r. A similar discussion was performed by Bell [7], but a completely different interpretation was put forward by this author.

6. Conclusions

At the light of the previous discussion and results, which are the consequence of trivial linear algebra arguments, as well as the application to EPR wavefunctions of the simplest among the quantum mechanical postulates, one can ask: where is the EPR paradox?

System **S(2)** functions act as coordinates of the EPR wavefunction representation and their norms can be transformed as probability weights for expectation value measurements. Of course, they are different for each operator chosen, but essentially constitute the *same* coordinate set, rotated into infinite dimensional space, as a new orthonormalized reference frame is chosen.

One can even expect that as an outcome of a measurement, an eigenvalue of some Hermitian operator appears, thus impliying that a simple relationship involving operator eigenvalues and probability coefficents hold.

No incoherence with quantum mechanical well known ideas, incompletness or faulty scientific realism appears as a consequence of the present analysis.

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